

PP-waves with Torsion - a Metric-affine Model for the Massless Neutrino

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Abstract In this paper we deal with quadratic metric–affine gravity, which we briefly introduce, explain and give historical and physical reasons for using this particular theory of gravity. We then introduce a generalisation of well known spacetimes, namely pp-waves. A classical pp-wave is a 4-dimensional Lorentzian spacetime which admits a nonvanishing parallel spinor field; here the connection is assumed to be Levi-Civita. This definition was generalised in our previous work to metric compatible spacetimes with torsion and used to construct new explicit vacuum solutions of quadratic metric–affine gravity, namely *generalised* pp-waves of parallel Ricci curvature. The physical interpretation of these solutions we propose in this article is that they represent a *conformally invariant metric–affine model for a massless elementary particle*. We give a comparison with the classical model describing the interaction of gravitational and massless neutrino fields, namely *Einstein–Weyl theory* and construct pp-wave type solutions of this theory. We point out that generalised pp-waves of parallel Ricci curvature are very similar to pp-wave type solutions of the Einstein–Weyl model and therefore propose that our generalised pp-waves of parallel Ricci curvature represent a metric–affine model for the massless neutrino.

Keywords quadratic metric–affine gravity · pp-waves · torsion · massless neutrino · Einstein–Weyl theory

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1 Introduction

The smallest departure from a Riemannian spacetime of Einstein's general relativity would consist of admitting *torsion* (8), arriving thereby at a Riemann–Cartan spacetime, and, furthermore, possible nonmetricity (11), resulting in a '*metric–affine*' spacetime. Metric–affine gravity is a natural generalisation of Einstein's general relativity, which is based on a spacetime with a Riemannian metric g of Lorentzian signature.

We consider spacetime to be a connected real 4-manifold M equipped with a Lorentzian metric g and an affine connection Γ . The 10 independent components of the (symmetric) metric tensor $g_{\mu\nu}$ and the 64 connection coefficients $\Gamma^\lambda_{\mu\nu}$ are the unknowns of our theory. Note that the characterisation of the spacetime manifold by an *independent* linear connection Γ initially distinguishes metric–affine gravity from general relativity. The connection incorporates the inertial properties of spacetime and it can be viewed, according to Hermann Weyl [72], as the guidance field of spacetime. The metric describes the structure of spacetime with respect to its spacio-temporal distance relations.

According to Hehl et al. in [33], in Einstein's general relativity the linear connection of its Riemannian spacetime is metric–compatible and symmetric. The symmetry of the Levi–Civita connection translates into the closure of infinitesimally small parallelograms. Already the transition from the flat gravity-free Minkowski spacetime to the Riemannian spacetime in Einstein's theory can locally be understood as a deformation process. The lifting of the constraints of metric–compatibility and symmetry yields nonmetricity and torsion, respectively. The continuum under consideration, here classical spacetime, is thereby assumed to have a non-trivial microstructure, similar to that of a liquid crystal or a dislocated metal or ferromagnetic material etc. It is gratifying to have the geometrical concepts of nonmetricity and torsion already arising in the (three-dimensional) continuum theory of lattice defects, see [39,40].

We define our action as

$$S := \int q(R) \quad (1)$$

where q is an $O(1,3)$ -invariant quadratic form on curvature R . Independent variation of the metric g and the connection Γ produces Euler–Lagrange equations which we will write symbolically as

$$\partial S / \partial g = 0, \quad (2)$$

$$\partial S / \partial \Gamma = 0. \quad (3)$$

Our objective is the study of the combined system of field equations (2), (3). This is a system of $10 + 64$ real nonlinear partial differential equations with $10 + 64$ real unknowns.

Our motivation comes from Yang–Mills theory. The Yang–Mills action for the affine connection is a special case of (1) with

$$q(R) = q_{\text{YM}}(R) := R^\kappa_{\lambda\mu\nu} R^\lambda_{\kappa}{}^{\mu\nu}. \quad (4)$$

With this choice of $q(R)$, equation (3) is the Yang–Mills equation for the affine connection, which was analysed by Yang [74]. The quadratic form q appearing in (1) is

a generalisation of (4). The general formula for q contains 16 different R^2 -terms with 16 coupling constants. This formula is given in Appendix B of [71]. An equivalent formula can be found in [18, 34].

Yang was looking for Riemannian solutions, so he specialised equation (3) to the Levi–Civita connection

$$\partial S / \partial \Gamma|_{\text{L-C}} = 0. \quad (5)$$

Here ‘specialisation’ means that one sets $\Gamma^\lambda_{\mu\nu} = \left\{ \begin{smallmatrix} \lambda \\ \mu\nu \end{smallmatrix} \right\}$ after the variation in Γ is carried out. It is known [68] that in this case for a generic 11-parameter action equation (5) reduces to

$$\nabla_\lambda Ric_{\kappa\mu} - \nabla_\kappa Ric_{\lambda\mu} = 0. \quad (6)$$

However, according to [71] for a generic 16-parameter action equation (5) reduces to

$$\nabla Ric = 0. \quad (7)$$

The field equations (6) and (7) are very much different, with (7) being by far more restrictive. In particular, Nordström–Thompson spacetimes (Riemannian spacetimes with $*R^* = +R$) satisfy (6) but do not necessarily satisfy (7). Note that the LHS of (6) is the Cotton tensor for metric compatible spacetimes with constant scalar curvature, see e.g. [15, 22]. Compare this to equation (8) from [71] and equation (27) from [53].

Let us here mention the contributions of C.N. Yang [74] and E.W. Mielke [46] who showed, respectively, that Einstein spaces satisfy equations (3) and (2). There is a substantial bibliography devoted to the study of the system (2), (3) in the special case (4) and one can get an idea of the historical development of the Yang–Mielke theory of gravity from [13, 19, 20, 50, 55, 64, 66, 67, 73].

The motivation for choosing a model of gravity which is purely quadratic in curvature is explained in Section 1 of [71]. The study of equations (2), (3) for specific purely quadratic curvature Lagrangians has a long history. Quadratic curvature Lagrangians were first discussed by Weyl [72], Pauli [54], Eddington [17] and Lanczos [42, 43, 44] in an attempt to include the electromagnetic field in Riemannian geometry.

The idea of using a purely quadratic action in General Relativity goes back to Hermann Weyl, as given at the end of his paper [72], where he argued that the most natural gravitational action should be quadratic in curvature and involve all possible invariant quadratic combinations of curvature, like the square of Ricci curvature, the square of scalar curvature, etc. Unfortunately, Weyl himself never afterwards pursued this analysis. Stephenson [64] looked at three different quadratic invariants: scalar curvature squared, Ricci curvature squared and the Yang–Mills Lagrangian (4) and varied with respect to the metric and the affine connection. He concluded that every equation arising from the above mentioned quadratic Lagrangians has the Schwarzschild solution and that the equations give the same results for the three ‘crucial tests’ of general relativity, i.e. the bending of light, the advance of the perihelion of Mercury and the red-shift.

Higgs [36] continued in a similar fashion to show that in scalar squared and Ricci squared cases, one set of equations may be transformed into field equations of the Einstein type with an arbitrary ‘cosmological constant’ in terms of the ‘new gauge-invariant metric’.

One can get more information and form an idea on the historical development of the quadratic metric–affine theory of gravity from [13, 19, 20, 36, 46, 50, 55, 64, 66, 67, 73, 74].

It should be noted that the action (1) contains only purely quadratic curvature terms, so it excludes the Einstein–Hilbert term (linear in curvature) and any terms quadratic in torsion (8) and nonmetricity (11). By choosing a purely quadratic curvature Lagrangian we are hoping to describe phenomena whose characteristic wavelength is sufficiently small and curvature sufficiently large.

We should also point out that the action (1) is conformally invariant, i.e. it does not change if we perform a Weyl rescaling of the metric $g \rightarrow e^{2f}g$, $f : M \rightarrow \mathbb{R}$, without changing the connection Γ .

In our previous work [53], we presented new non-Riemannian solutions of the field equations (2), (3). These new solutions were to be constructed explicitly and the construction turned out to be very similar to the classical construction of a pp-wave, only with torsion. This paper aims to give additional information on these spacetimes, provide their physical interpretation, additional calculations and future possible applications.

The paper has the following structure. In section 3 we recall basic facts about pp-waves with torsion, in subsection 3.1 we provide information about classical pp-waves, in subsection 3.2 we recall the way pp-waves were generalised in [53] and list the properties of these spacetimes with torsion and in subsection 3.3 we present the pp-waves spinor formalism. In section 4 we present our attempt at giving a physical interpretation to the solutions of the field equations from [53]. In subsection 4.1 we provide a reminder on the classical model describing the interaction of gravitational and massless neutrino fields (Einstein–Weyl theory), while in subsection 4.1.1 we present a brief review of known solutions of this theory. In subsection 4.2 we present our pp-wave type solutions of this theory. In subsection 4.3 we compare the Einstein–Weyl solutions to our conformally invariant solutions. Finally, appendix A provides the spinor formalism used throughout our work, appendix B gives detailed calculations involved in comparing our solutions to Einstein–Weyl theory and appendix C provides a correction of a mistake found in our previous work [53] and gives the motivation for future work.

2 Notation

Our notation follows [37, 52, 53, 68, 71]. In particular, we denote local coordinates by x^μ , $\mu = 0, 1, 2, 3$, and write $\partial_\mu := \partial/\partial x^\mu$. We define torsion as

$$T^\lambda{}_{\mu\nu} := \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}. \quad (8)$$

The irreducible pieces of torsion are, following [68],

$$T^{(1)} = T - T^{(2)} - T^{(3)}, \quad T^{(2)}{}_{\lambda\mu\nu} = g_{\lambda\mu}v_\nu - g_{\lambda\nu}v_\mu, \quad T^{(3)} = *w, \quad (9)$$

where

$$v_\nu = \frac{1}{3}T^\lambda{}_{\lambda\nu}, \quad w_\nu = \frac{1}{6}\sqrt{|\det g|}T^{\kappa\lambda\mu}\varepsilon_{\kappa\lambda\mu\nu}. \quad (10)$$

The pieces $T^{(1)}$, $T^{(2)}$ i $T^{(3)}$ are called *tensor torsion*, *trace torsion*, and *axial torsion* respectively. We say that our connection Γ is metric compatible if $\nabla g \equiv 0$. The interval is $ds^2 := g_{\mu\nu} dx^\mu dx^\nu$. Given a scalar function $f : M \rightarrow \mathbb{R}$ we write for brevity $\int f := \int_M f \sqrt{|\det g|} dx^0 dx^1 dx^2 dx^3$, $\det g := \det(g_{\mu\nu})$. We define *nonmetricity* by

$$Q_{\mu\alpha\beta} := \nabla_\mu g_{\alpha\beta}. \quad (11)$$

We use the term ‘parallel’ to describe the situation when the covariant derivative of some spinor or tensor field is identically zero. We do not assume that our spacetime admits a (global) spin structure, cf. Section 11.6 of [47]. In fact, our only topological assumption is connectedness. This does not prevent us from defining and parallel transporting spinors or tensors locally.

3 PP-waves With Torsion

In this section, where we mostly follow the exposition from [53], we provide background information about *pp-waves*, starting with the notion of a classical *pp-wave*, then introducing a generalisation with the addition of torsion and lastly presenting the particular spinor formalism of *pp-waves*.

3.1 Classical pp-waves

PP-waves are well known spacetimes in general relativity, first discovered by Brinkmann [10] in 1923, and subsequently rediscovered by several authors, for example Peres [57] in 1959. There are differing views on what the ‘pp’ stands for. According to Griffiths [23] and Kramer et al. [38] ‘pp’ is an abbreviation for ‘plane-fronted gravitational waves with parallel rays’. See e.g. [1, 5, 8, 10, 21, 23, 38, 48, 49, 51, 53, 57, 58, 70, 71] for more information on *pp-waves* and *pp-wave* type solutions of metric-affine gravity.

Definition 1 A *pp-wave* is a Riemannian spacetime which admits a nonvanishing parallel spinor field.

It was only relatively recently discovered in [71] that *pp-waves* of parallel Ricci curvature are solutions of (2), (3). This section closely follows Section 3 from [53] in exposition and we only present the the most important facts about these well known spacetimes. The nonvanishing parallel spinor field appearing in the definition of *pp-waves* will be denoted throughout this paper by

$$\chi = \chi^a$$

and we assume this spinor field to be *fixed*. Put

$$l^\alpha := \sigma^\alpha_{ab} \chi^a \bar{\chi}^b \quad (12)$$

where the σ^α are Pauli matrices¹. Then l is a nonvanishing parallel real null vector field. Now we define the real scalar function

$$\varphi : M \rightarrow \mathbb{R}, \quad \varphi(x) := \int l \cdot dx. \quad (13)$$

This function is called the *phase*. It is defined uniquely up to the addition of a constant and possible multi-valuedness resulting from a nontrivial topology of the manifold. Put

$$F_{\alpha\beta} := \sigma_{\alpha\beta ab} \chi^a \chi^b \quad (14)$$

where the $\sigma_{\alpha\beta}$ are ‘second order Pauli matrices’ (37), (60). Then F is a nonvanishing parallel complex 2-form with the additional properties $*F = \pm iF$ and $\det F = 0$. It can be written as

$$F = l \wedge m \quad (15)$$

where m is a complex vector field satisfying $m_\alpha m^\alpha = l_\alpha m^\alpha = 0$, $m_\alpha \bar{m}^\alpha = -2$. It is known, see [2, 11], that Definition 1 is equivalent to the following

Definition 2 A *pp-wave* is a Riemannian spacetime whose metric can be written locally in the form

$$ds^2 = 2 dx^0 dx^3 - (dx^1)^2 - (dx^2)^2 + f(x^1, x^2, x^3) (dx^3)^2 \quad (16)$$

in some local coordinates (x^0, x^1, x^2, x^3) .

The remarkable property of the metric (16) is that the corresponding curvature tensor R is linear in f :

$$R_{\alpha\beta\gamma\delta} = -\frac{1}{2} (l \wedge \partial)_{\alpha\beta} (l \wedge \partial)_{\gamma\delta} f \quad (17)$$

where $(l \wedge \partial)_{\alpha\beta} := l_\alpha \partial_\beta - \partial_\alpha l_\beta$. The advantage of Definition 2 is that it gives an explicit formula for the metric of a pp-wave. Its disadvantage is that it relies on a particular choice of local coordinates in each coordinate patch. The choice of local coordinates in which the pp-metric assumes the form (16) is not unique. We will restrict our choice to those coordinates in which

$$\chi^a = (1, 0), \quad l^\mu = (1, 0, 0, 0), \quad m^\mu = (0, 1, \mp i, 0). \quad (18)$$

With such a choice formula (13) reads $\varphi(x) = x^3 + \text{const}$. Formula (17) can be rewritten in invariant form

$$R = -\frac{1}{2} (l \wedge \nabla) \otimes (l \wedge \nabla) f \quad (19)$$

where $l \wedge \nabla := l \otimes \nabla - \nabla \otimes l$. The curvature of a pp-wave has the following irreducible pieces: (symmetric) trace-free Ricci and Weyl. Ricci curvature is proportional to $l \otimes l$ whereas Weyl curvature is a linear combination of $\text{Re}((l \wedge m) \otimes (l \wedge m))$

¹ See appendix A for our general spinor formalism and section 3.3 for spinor formalism for pp-waves

and $\text{Im}((l \wedge m) \otimes (l \wedge m))$. In our special local coordinates (16), (18), we can express these as

$$\text{Ric} = \frac{1}{2}(f_{11} + f_{22})l \otimes l, \quad (20)$$

$$\mathcal{W} = \sum_{j,k=1}^2 w_{jk}(l \wedge m_j) \otimes (l \wedge m_k), \quad (21)$$

where $m_1 = \text{Re}(m)$, $m_2 = \text{Im}(m)$, $f_{\alpha\beta} := \partial_\alpha \partial_\beta f$ and w_{jk} are real scalars given by $w_{11} = \frac{1}{4}(-f_{11} + f_{22})$, $w_{12} = \pm \frac{1}{2}f_{12}$, $w_{22} = -w_{11}$, $w_{21} = w_{12}$.

Remark 1 Note that the Cotton tensor of classical pp-waves with parallel Ricci curvature vanishes, as classical pp-waves are metric compatible spacetimes with zero scalar curvature. In the theory of conformal spaces the main geometrical objects to be analysed are the Weyl and the Cotton tensors, see [22]. It is well known that for conformally flat spaces the Weyl tensor has to vanish and consequently the Cotton tensor has to vanish too. Note that the Cotton tensor is only conformally invariant in three dimensions.

3.2 Generalised pp-waves

One natural way of generalising the concept of a classical pp-wave is simply to extend Definition 1 to general metric compatible spacetimes, i.e. spacetimes whose connection is not necessarily Levi-Civita. However, this gives a class of spacetimes which is too wide and difficult to work with. We choose to extend the classical definition in a more special way better suited to the study of the system of field equations (2), (3).

Consider the polarized Maxwell equation²

$$*dA = \pm idA \quad (22)$$

in a classical pp-space. Here A is the unknown complex vector field. The motivation for calling equation (22) the polarized Maxwell equation comes from the fact that any solution of (22) is a solution of the Maxwell equation $\delta du = 0$, see [37]. We seek plane wave solutions of (22). These can be written down explicitly:

$$A = h(\varphi)m + k(\varphi)l. \quad (23)$$

Here l and m are the vector fields defined in section 3.1, $h, k : \mathbb{R} \rightarrow \mathbb{C}$ are arbitrary functions, and φ is the phase (13).

Definition 3 A *generalised pp-wave* is a metric compatible spacetime with pp-metric and torsion

$$T := \frac{1}{2}\text{Re}(A \otimes dA) \quad (24)$$

where A is a vector field of the form (23).

² See in particular [37] as well as [51, 53, 68, 69, 70, 71]

We list below the main properties of generalised pp-waves. Note that here and further on we denote by $\{\nabla\}$ the covariant derivative with respect to the Levi-Civita connection which should not be confused with the full covariant derivative ∇ incorporating torsion.

The curvature of a generalised pp-wave is

$$R = -\frac{1}{2}(l \wedge \{\nabla\}) \otimes (l \wedge \{\nabla\})f + \frac{1}{4}\text{Re}((h^2)''(l \wedge m) \otimes (l \wedge m)). \quad (25)$$

and the torsion of a generalised pp-wave is

$$T = \text{Re}((a l + b m) \otimes (l \wedge m)), \quad (26)$$

where

$$a := \frac{1}{2}h'(\varphi)k(\varphi), \quad b := \frac{1}{2}h'(\varphi)h(\varphi).$$

Torsion can be written down even more explicitly in the following form

$$T = \sum_{j,k=1}^2 t_{jk} m_j \otimes (l \wedge m_k) + \sum_{j=1}^2 t_j l \otimes (l \wedge m_j), \quad (27)$$

where

$$t_{11} = -t_{22} = \frac{1}{2}\text{Re}(b), \quad t_{12} = t_{21} = -\frac{1}{2}\text{Im}(b), \quad t_1 = \frac{1}{2}\text{Re}(a), \quad t_2 = -\frac{1}{2}\text{Im}(a),$$

$m_1 = \text{Re}(m)$, $m_2 = \text{Im}(m)$ and a and b at the same functions of the phase φ appearing in equation (26).

Remark 2 From equation (27) we can clearly see that torsion has 4 independent non-zero components.

In the beginning of section 3.1 we introduced the spinor field χ satisfying $\{\nabla\}\chi = 0$. It becomes clear that this spinor field also satisfies $\nabla\chi = 0$.

Lemma 1 *The generalised pp-wave and the underlying classical pp-wave admit the same nonvanishing parallel spinor field.*

Proof To see that $\nabla\chi = 0$, we look at the only remaining torsion generated term from formula (63) for the spinor connection coefficients $\Gamma^a_{\mu b}$, namely

$$\nabla_\mu \chi^a = \{\nabla\}_\mu \chi^a + \frac{1}{4}\sigma^{\alpha a \dot{c}} T_{\mu \alpha \beta} \sigma^{\beta}_{\dot{c} b} \chi^b.$$

In view of equation (26), it is sufficient to show that the term involving $(l \wedge m)$ contracted with the Pauli matrices gives zero. To see this, we rewrite the term in the following form

$$\begin{aligned} \sigma^{\alpha a \dot{c}} (l \wedge m)_{\alpha \beta} \sigma^{\beta}_{\dot{c} b} &= \frac{1}{2}(l \wedge m)_{\alpha \beta} (\sigma^{\alpha a \dot{c}} \sigma^{\beta}_{\dot{c} b} - \sigma^{\beta a \dot{c}} \sigma^{\alpha}_{\dot{c} b}), \\ &= (l \wedge m)_{\alpha \beta} \sigma^{\alpha \beta a}_{\dot{c} b} = 0, \end{aligned}$$

which can be checked directly using our local coordinates (16), (18) and the second order Pauli matrices (37), i.e.

$$-\sigma^{13}_{ab} - i\sigma^{23}_{ab} + \sigma^{31}_{ab} + i\sigma^{31}_{ab} = 0.$$

Hence, $\nabla\chi = 0$. \square

Remark 3 In view of Lemma 1, it is clear that both the generalised pp-wave and the underlying classical pp-wave admit the same nonvanishing parallel real null vector field l and the same nonvanishing parallel complex 2-form (14), (15).

Lemma 2 *The torsion (24) of a generalised pp-wave is purely tensor³, i.e.*

$$T^\alpha{}_{\alpha\gamma} = 0, \quad \varepsilon_{\alpha\beta\gamma\delta} T^{\alpha\beta\gamma} = 0.$$

Proof The first equation $T^\alpha{}_{\alpha\gamma} = 0$ follows directly from equation (26) and the fact that we have that $l_\alpha l^\alpha = m_\alpha m^\alpha = 0$, see section 3.1. The second equation $\varepsilon_{\alpha\beta\gamma\delta} T^{\alpha\beta\gamma} = 0$ follows from $*F = \pm iF$, see equations (14), (15). We then have

$$\varepsilon_{\alpha\beta\gamma\delta} (l \wedge m)^{\beta\gamma} = Z (l \wedge m)_{\alpha\delta},$$

where $Z \in \mathbb{C}$ is some constant. Then using the formula for torsion (26) we have

$$\varepsilon_{\alpha\beta\gamma\delta} T^{\alpha\beta\gamma} = \text{Re}(Z(a l + b m)^\alpha (l \wedge m)_{\alpha\delta}) = 0,$$

using the same argument as before, i.e. the fact that $l_\alpha l^\alpha = m_\alpha m^\alpha = 0$. \square

Examination of formula (25) for the curvature of a generalised pp-wave reveals the following remarkable properties of generalised pp-waves:

- The curvatures generated by the Levi-Civita connection and torsion simply add up (compare formulae (19) and (25)).
- The second term in the RHS of (25) is purely Weyl. Consequently, the Ricci curvature of a generalised pp-wave is completely determined by the pp-metric.
- Clearly, generalised pp-waves have the same non-zero irreducible pieces of curvature as classical pp-waves, namely symmetric trace-free Ricci and Weyl. Using special local coordinates (16), (18), these can be expressed explicitly as

$$\begin{aligned} Ric &= \frac{1}{2}(f_{11} + f_{22})l \otimes l, \\ \mathcal{W} &= \sum_{j,k=1}^2 w_{jk} (l \wedge m_j) \otimes (l \wedge m_k), \end{aligned}$$

where $m_1 = \text{Re}(m)$, $m_2 = \text{Im}(m)$, $f_{\alpha\beta} := \partial_\alpha \partial_\beta f$ and w_{jk} are real scalars given by

$$\begin{aligned} w_{11} &= \frac{1}{4}[-f_{11} + f_{22} + \text{Re}((h^2)'')], \quad w_{22} = -w_{11}, \\ w_{12} &= \pm \frac{1}{2}f_{12} - \frac{1}{4}\text{Im}((h^2)''), \quad w_{21} = w_{12}. \end{aligned}$$

Compare these to the corresponding equations (20) and (21) for classical pp-waves.

³ Only the $T^{(1)}$, or ‘tensor torsion’ irreducible piece of torsion is non-zero, see equations (9), (10).

- The curvature of a generalised pp-wave has all the usual symmetries of curvature in the Riemannian case, that is,

$$R_{\kappa\lambda\mu\nu} = R_{\mu\nu\kappa\lambda}, \quad (28)$$

$$\varepsilon^{\kappa\lambda\mu\nu} R_{\kappa\lambda\mu\nu} = 0, \quad (29)$$

$$R_{\kappa\lambda\mu\nu} = -R_{\lambda\kappa\mu\nu}, \quad (30)$$

$$R_{\kappa\lambda\mu\nu} = -R_{\kappa\lambda\nu\mu}. \quad (31)$$

Of course, (31) is true for any curvature whereas (30) is a consequence of metric compatibility. Also, (30) follows from (28) and (31).

- The second term in the RHS of (23) is pure gauge in the sense that it does not affect curvature (25). It does, however, affect torsion (24).
- The Ricci curvature of a generalised pp-wave is zero if and only if

$$f_{11} + f_{22} = 0 \quad (32)$$

and the Weyl curvature is zero if and only if

$$f_{11} - f_{22} = \operatorname{Re}((h^2)'''), \quad f_{12} = \pm \frac{1}{2} \operatorname{Im}((h^2)'''). \quad (33)$$

Here we use special local coordinates (16), (18) and denote $f_{\alpha\beta} := \partial_\alpha \partial_\beta f$.

- The curvature of a generalised pp-wave is zero if and only if we have both (32) and (33). Clearly, for any given function h we can choose a function f such that $R = 0$: this f is a quadratic polynomial in x^1, x^2 with coefficients depending on x^3 . Thus, as a spin-off, we get a class of examples of Weitzenböck spaces ($T \neq 0, R = 0$).

3.3 Spinor formalism for generalised pp-waves

In this section we provide the particular spinor formalism for generalised pp-waves. For the pp-metric (16) we choose Pauli matrices

$$\begin{aligned} \sigma^0_{ab} &= \begin{pmatrix} 1 & 0 \\ 0 & -f \end{pmatrix}, \quad \sigma^1_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \sigma^2_{ab} &= \begin{pmatrix} 0 & \mp i \\ \pm i & 0 \end{pmatrix}, \quad \sigma^3_{ab} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}. \end{aligned} \quad (34)$$

Our two choices of Pauli matrices differ by orientation. When dealing with a classical pp-wave the choice of orientation of Pauli matrices does not really matter, but for a generalised pp-wave it is convenient to choose orientation of Pauli matrices in agreement with the sign in (22) and (41) as this simplifies the resulting formulae.

Remark 4 In the case $f = 0$, formulae (34) do not turn into the Minkowski space Pauli matrices, since we write the metric in the form (16). This is a matter of convenience in calculations.

Remark 5 Note that we could have chosen a different set of Pauli matrices σ^{α}_{ab} in (34), namely with the opposite sign in every Pauli matrix, as they are a basis in the real vector space of Hermitian 2×2 matrices σ_{ab} satisfying $\sigma^{\alpha}_{ab}\sigma^{\beta cb} + \sigma^{\beta}_{ab}\sigma^{\alpha cb} = 2g^{\alpha\beta}\delta_a^c$, defined uniquely up to a Lorentz transformation. See appendix A for more on our chosen spinor formalism.

Now we want to describe the spinor connection coefficients $\Gamma^a_{\mu b}$, see formula (63) in appendix A. For a generalised pp-wave, the non-zero coefficients of $\Gamma^a_{\mu b}$ are

$$\Gamma^1_{12} = \frac{1}{2}hh', \quad \Gamma^1_{22} = \mp \frac{i}{2}hh', \quad \Gamma^1_{32} = \frac{1}{2} \left(\frac{\partial f}{\partial x^1} \pm i \frac{\partial f}{\partial x^2} \right) - \frac{1}{2}kh'.$$

Here we use special local coordinates (16), (18) and Pauli matrices (34). Note that with our choice of Pauli matrices the signs in formulae (34) and (61) agree.

Since by Lemma 2 the torsion of a generalised pp-wave is purely tensor, the massless Dirac equation (65), (66) (also called Weyl's equation) takes the form

$$\sigma^{\mu}_{ab} \nabla_{\mu} \xi^a = 0, \quad (35)$$

or equivalently

$$\sigma^{\mu}_{ab} \{\nabla\}_{\mu} \xi^a = 0, \quad (36)$$

see appendix B for more on the massless Dirac equation.

Remark 6 In view of equations (35) and (36), it is easy to see that $\chi F(\varphi)$ is a solution of the massless Dirac equation. Here F is an arbitrary function of the phase (13) and χ is the parallel spinor introduced in section 3.1.

We also provide the explicit formulae for the 'second order Pauli matrices' (60) for the pp-metric (16).

$$\begin{aligned} \sigma^{01}_{ab} &= \begin{pmatrix} 1 & 0 \\ 0 & f \end{pmatrix}, \quad \sigma^{02}_{ab} = \begin{pmatrix} \mp i & 0 \\ 0 & \pm if \end{pmatrix}, \quad \sigma^{03}_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \sigma^{12}_{ab} &= \begin{pmatrix} 0 & \mp i \\ \mp i & 0 \end{pmatrix}, \quad \sigma^{13}_{ab} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \quad \sigma^{23}_{ab} = \begin{pmatrix} 0 & 0 \\ 0 & \pm 2i \end{pmatrix}. \end{aligned} \quad (37)$$

Note that as the second order Pauli matrices $\sigma^{\alpha\beta}_{ab}$ are antisymmetric over the tensor indices, i.e. $\sigma^{\alpha\beta}_{ab} = -\sigma^{\beta\alpha}_{ab}$, we only give the independent non-zero terms.

4 Physical interpretation of generalised pp-waves

It was shown in our previous work [53] that using the generalised pp-waves described in section 3.2 we can construct new vacuum solutions of quadratic metric-affine gravity. The main result of [53] is the following

Theorem 1 *Generalised pp-waves of parallel Ricci curvature are solutions of the system of equations (2), (3).*

The observation that one can construct vacuum solutions of quadratic metric-affine gravity in terms of pp-waves is a recent development. The fact that classical pp-spaces of parallel Ricci curvature are solutions was first pointed out in [69, 70, 71].

Remark 7 There is a slight error in our calculations of the explicit form of the field equations from [53], which in no way influences the main result. The error was noticed in producing [52], where the generalised version of the explicit field equations can be found, see Appendix C and for more details.

In this section we attempt to give a physical interpretation of the vacuum solutions of field equations (2), (3) obtained in [53], the notation and (partially) exposition from which we follow here. This topic was also explored very briefly and without explicit calculations in our research review paper [51]. As noted in [53], the following two classes of Riemannian spacetimes are solutions of our field equations:

- Einstein spaces ($Ric = \Lambda g$), and
- classical pp-spaces of parallel Ricci curvature.

In general relativity, Einstein spaces are an accepted mathematical model for vacuum. However, classical pp-spaces of parallel Ricci curvature do not have an obvious physical interpretation. This section gives an attempt at understanding whether our newly constructed spacetimes are of mathematical or physical significance.

Our analysis of vacuum solutions of quadratic metric-affine gravity shows that classical pp-spaces of parallel Ricci curvature should not be viewed on their own. They are a particular (degenerate) representative of a wider class of solutions, namely, generalised pp-spaces of parallel Ricci curvature. Indeed, the curvature of a generalised pp-space is a sum of two curvatures: the curvature

$$-\frac{1}{2}(l \wedge \{\mathbb{V}\}) \otimes (l \wedge \{\mathbb{V}\})f \quad (38)$$

of the underlying classical pp-space and the curvature

$$\frac{1}{4}\text{Re}((h^2)''(l \wedge m) \otimes (l \wedge m)) \quad (39)$$

generated by a torsion wave traveling over this classical pp-space. Our torsion and corresponding curvature (39) are waves traveling at speed of light because h and k are functions of the phase φ which plays the role of a null coordinate, $g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi = 0$. The underlying classical pp-space of parallel Ricci curvature can now be viewed as the ‘gravitational imprint’ created by a wave of some massless matter field. Such a situation occurs in Einstein–Maxwell theory⁴ and Einstein–Weyl theory⁵. The difference with our model is that Einstein–Maxwell and Einstein–Weyl theories contain the gravitational constant which dictates a particular relationship between the strengths of the fields in question, whereas our model is conformally invariant and the amplitudes of the two curvatures (38) and (39) are totally independent.

⁴ A classical model describing the interaction of gravitational and electromagnetic fields

⁵ A classical model describing the interaction of gravitational and massless neutrino fields

In the remainder of this subsection we outline an argument in favour of interpreting our torsion wave (24), (23) as a mathematical model for some massless particle.

We base our interpretation on the analysis of the curvature (39) generated by our torsion wave. Examination of formula (39) indicates that it is more convenient to deal with the complexified curvature

$$\mathfrak{R} := r(l \wedge m) \otimes (l \wedge m) \quad (40)$$

where $r := \frac{1}{4}(h^2)''$ (this r is a function of the phase φ); note also that complexification is in line with the traditions of quantum mechanics. Our complex curvature is polarized,

$$*\mathfrak{R} = \mathfrak{R}^* = \pm i\mathfrak{R}, \quad (41)$$

and purely Weyl, hence it is equivalent to a (symmetric) rank 4 spinor ω . The relationship between \mathfrak{R} and ω is given by the formula

$$\mathfrak{R}_{\alpha\beta\gamma\delta} = \sigma_{\alpha\beta ab} \omega^{abcd} \sigma_{\gamma\delta cd} \quad (42)$$

where the $\sigma_{\alpha\beta}$ are ‘second order Pauli matrices’ (60). Resolving (42) with respect to ω we get, in view of (14), (15), (40),

$$\omega = \xi \otimes \xi \otimes \xi \otimes \xi \quad (43)$$

where

$$\xi := r^{1/4} \chi \quad (44)$$

and χ is the spinor field introduced in the beginning of section 3.1.

Formula (43) shows that our rank 4 spinor ω has additional algebraic structure: it is the 4th tensor power of a rank 1 spinor ξ . Consequently, the complexified curvature generated by our torsion wave is completely determined by the rank 1 spinor field ξ .

We claim that the spinor field (44) satisfies the massless Dirac equation, see (35) or (36). Indeed, as χ is parallel checking that ξ satisfies the massless Dirac equation reduces to checking that $(r^{1/4})' \sigma_{ab}^{\mu} l_{\mu} \chi^a = 0$. The latter is established by direct substitution of the explicit formula (12) for l .

4.1 Einstein–Weyl field equations

In this section we aim to provide a reminder of Einstein–Weyl theory and the field equations arising from this classical model describing the interaction of gravitational and massless neutrino fields, then to provide pp-wave type solutions within this model, provide the previously known solutions of this type and, finally, to compare them to the pp-wave type solutions of our conformally invariant metric–affine model of gravity from [53].

In Einstein–Weyl theory the action is given by

$$S_{EW} := 2i \int \left(\xi^a \sigma_{ab}^{\mu} (\{\nabla\}_{\mu} \bar{\xi}^b) - (\{\nabla\}_{\mu} \xi^a) \sigma_{ab}^{\mu} \bar{\xi}^b \right) + k \int \mathcal{R}, \quad (45)$$

where the constant k can be chosen so that the non-relativistic limit yields the usual form of Newton’s gravity law. According to Brill and Wheeler [9], $k = \frac{c^4}{16\pi G}$, where G is the gravitational constant.

Remark 8 Note that in Einstein–Weyl theory the connection is assumed to be Levi-Civita, so we only vary the action (45) with respect to the metric and the spinor.

We obtain the well known Einstein–Weyl field equations

$$\frac{\delta S_{EW}}{\delta g} = 0, \quad (46)$$

$$\frac{\delta S_{EW}}{\delta \xi} = 0. \quad (47)$$

The first term of the action S depends on the spinor ξ and the metric g while the second depends on the metric g only. Hence the formal variation of the action (45) with respect to the spinor just yields the massless Dirac equation, see appendix B.

The variation with respect to the metric is somewhat more complicated as both terms of the action (45) depend on the metric g . The variation of the first term of the action with respect to the metric yields the energy momentum tensor of the Weyl action (64), i.e.

$$\begin{aligned} E^{\mu\nu} = & \frac{i}{2} \left[\sigma^{\nu}_{ab} \left(\bar{\xi}^b \{\nabla\}^{\mu} \xi^a - \xi^a \{\nabla\}^{\mu} \bar{\xi}^b \right) + \sigma^{\mu}_{ab} \left(\bar{\xi}^b \{\nabla\}^{\nu} \xi^a - \xi^a \{\nabla\}^{\nu} \bar{\xi}^b \right) \right] \\ & + i \left(\xi^a \sigma^{\eta}_{ab} (\{\nabla\}_{\eta} \bar{\xi}^b) g^{\mu\nu} - (\{\nabla\}_{\eta} \xi^a) \sigma^{\eta}_{ab} \bar{\xi}^b g^{\mu\nu} \right). \end{aligned} \quad (48)$$

Note that the energy momentum tensor is not a priori trace-free. Please see appendix B.1 for the detailed derivation of formula (48).

Variation with respect to the metric of the Einstein–Hilbert term of the action yields

$$\delta \int \mathcal{R} = - \int (Ric^{\mu\nu} - \frac{1}{2} \mathcal{R} g^{\mu\nu}) \delta g_{\mu\nu},$$

which is a straightforward calculation, see e.g. Landau and Lifshitz [45]. Hence we get the explicit representation of the Einstein–Weyl field equations (46), (47):

$$\begin{aligned} & \frac{i}{2} \left[\sigma^{\nu}_{ab} \left(\bar{\xi}^b \{\nabla\}^{\mu} \xi^a - \xi^a \{\nabla\}^{\mu} \bar{\xi}^b \right) + \sigma^{\mu}_{ab} \left(\bar{\xi}^b \{\nabla\}^{\nu} \xi^a - \xi^a \{\nabla\}^{\nu} \bar{\xi}^b \right) \right] \\ & + i \left(\xi^a \sigma^{\eta}_{ab} (\{\nabla\}_{\eta} \bar{\xi}^b) g^{\mu\nu} - (\{\nabla\}_{\eta} \xi^a) \sigma^{\eta}_{ab} \bar{\xi}^b g^{\mu\nu} \right) \\ & - k Ric^{\mu\nu} + \frac{k}{2} \mathcal{R} g^{\mu\nu} = 0, \quad (49) \\ & \sigma^{\mu}_{ab} \{\nabla\}_{\mu} \xi^a = 0. \quad (50) \end{aligned}$$

Remark 9 When the equation (50) is satisfied, we have that the energy–momentum tensor (48) is trace free and the second line of (49) vanishes, see e.g. end of section 2 of Griffiths and Newing [25].

4.1.1 Known solutions of Einstein–Weyl theory

In one of the early works on this subject, Griffiths and Newing [24] show how the solutions of Einstein–Weyl equations can be constructed and present five examples of solutions and a later work by the same authors [25] presents a more general solution of Kundt’s class. Audretsch and Graf [4] derive a differential equation representing radiation solutions of the general relativistic Weyl’s equation and study the corresponding energy-momentum tensor and they present an exact solution of Einstein–Weyl equations in the form of pp-waves. Audretsch [3] continues to study the asymptotic behaviour of the neutrino energy-momentum tensor in curved space-time with the sole aid of generally covariant assumptions about the nature of the Weyl field and the author shows that these Weyl fields behave asymptotically like neutrino radiation.

Griffiths [26] expanded on his previous work and this paper is of particular interest to us as in section 5 of [26] the author presents solutions whose metric is the pp-wave metric (16) and the author presents a condition on the function f from the pp-metric (16). Griffiths [27] identifies a class of neutrino fields with zero energy momentum tensor and stipulates that these spacetimes may also be interpreted as describing gravitational waves. Collinson and Morris [14] showed that these could be either pp-waves or Robinson–Trautman type N solutions presented in [24]. Subsequently these were called ‘ghost neutrinos’ by Davis and Ray in [16].

Kuchowicz and Żebrowski [41] expand on the work on ghost neutrinos trying to resolve this anomaly by considering non-zero torsion in the framework of Einstein–Cartan theory. Griffiths [28] also considers the possibility of non-zero torsion and in a more general work [29] he showed that neutrino fields in Einstein–Cartan theory must have metrics that belong to the family of solutions of Kundt’s class, which include the pp-waves. Singh and Griffiths [62] corrected several mistakes from [29] and showed that neutrino fields in Einstein–Cartan theory also include the Robinson–Trautman type N solutions and that any solution of the Einstein–Weyl equations in general relativity has a corresponding solution in Einstein–Cartan theory. Thus pp-wave type solutions of Einstein–Weyl equations have corresponding solutions in Einstein–Cartan theory. This paper was one of the main inspirations behind the result in section 4.2.

4.2 PP–wave type solutions of Einstein–Weyl theory

The aim of this section is to point out the fact that the nonlinear system of equations (49), (50) has solutions in the form of pp-waves. Throughout this section we use the set of local coordinates (16), (18) and Pauli matrices (34). We now present a class of explicit solutions of (49), (50) where the metric g is in the form of a pp-metric and the spinor ξ as in (44). As shown in the section 4, the spinor (44) satisfies the equation (50). In the setting of a pp-space scalar curvature vanishes and as the spinor χ appearing in formula (44) is parallel, in view of Remark 9 equation (49) becomes

$$\frac{i}{2}\sigma^{\nu}{}_{ab}\left(\bar{\xi}^b\{\nabla\}^{\mu}\xi^a - \xi^a\{\nabla\}^{\mu}\bar{\xi}^b\right) + \frac{i}{2}\sigma^{\mu}{}_{ab}\left(\bar{\xi}^b\{\nabla\}^{\nu}\xi^a - \xi^a\{\nabla\}^{\nu}\bar{\xi}^b\right) - kRic^{\mu\nu} = 0. \quad (51)$$

We now need to determine under what condition the equation (51) is satisfied. In our local coordinates, we have

$$Ric = \left(\frac{1}{2} \frac{\partial^2 f}{\partial (x^1)^2} + \frac{1}{2} \frac{\partial^2 f}{\partial (x^2)^2} \right) (l \otimes l). \quad (52)$$

Substituting formulae (44), (52) into equation (51), and using the fact that the spinor χ is parallel, we obtain the equality

$$i(\sigma^{\nu}_{ab} l^{\mu} + \sigma^{\mu}_{ab} l^{\nu}) \left((r^{1/4})' \overline{r^{1/4}} - r^{1/4} \overline{(r^{1/4})'} \right) \chi^a \overline{\chi}^b = \frac{1}{2} k l^{\mu} l^{\nu} \left(\frac{\partial^2 f}{\partial (x^1)^2} + \frac{\partial^2 f}{\partial (x^2)^2} \right).$$

Since we know that $\sigma^{\mu}_{ab} \chi^a \overline{\chi}^b = l^{\mu}$, we obtain the condition for a pp-wave type solution of the Einstein–Weyl model

$$\frac{1}{2} \frac{\partial^2 f}{\partial (x^1)^2} + \frac{1}{2} \frac{\partial^2 f}{\partial (x^2)^2} = \frac{i}{k} \left((r^{1/4})' \overline{r^{1/4}} - r^{1/4} \overline{(r^{1/4})'} \right). \quad (53)$$

Thus, the complex valued function r of one real variable x^3 can be chosen arbitrarily and it uniquely determines the RHS of (53). From (53) one recovers the pp-metric by solving Poisson's equation.

4.3 Comparison of metric–affine and Einstein–Weyl solutions

To make our comparison clearer, let us compare these models in the case of monochromatic solutions of both models using local coordinates (16), (18) and Pauli matrices (34).

4.3.1 Monochromatic metric–affine solutions

In the case of the metric–affine model, from Theorem 1 we know that generalised pp-waves of parallel Ricci curvature are solutions of the equations (2), (3). Whether we are viewing monochromatic solutions or not, the condition on the solution of the model remains unchanged, namely Ricci curvature (52) has to be parallel. In our special local coordinates the condition of parallel Ricci curvature is written as

$$\frac{1}{2} \frac{\partial^2 f}{\partial (x^1)^2} + \frac{1}{2} \frac{\partial^2 f}{\partial (x^2)^2} = C, \quad (54)$$

where C is an arbitrary real constant. However, the construction of torsion simplifies in the monochromatic case. Namely, we can choose the function h of the phase (13) so that the plane wave (23) becomes

$$A = \frac{ic^2}{2a} e^{2i(ax^3+b)} m,$$

where $a, b, c \in \mathbb{R}$, $a \neq 0$. Torsion (24) then takes the form

$$T = -\frac{c^4}{4a} \operatorname{Re} \left(i e^{4i(ax^3+b)} m \otimes (l \wedge m) \right).$$

Hence the complexified curvature (40) generated by the torsion wave becomes

$$\mathfrak{R} = c^4 e^{4i(ax^3+b)} (l \wedge m) \otimes (l \wedge m),$$

and r from (40) becomes

$$r = c^4 e^{4i(ax^3+b)}.$$

The spinor ξ from (44) is explicitly given by

$$\xi = c \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i(ax^3+b)}. \quad (55)$$

4.3.2 Monochromatic Einstein–Weyl solutions

Let us now look for monochromatic solutions in Einstein–Weyl theory. We take the spinor field as in formula (55) in which case condition (53) simplifies to

$$\frac{1}{2} \frac{\partial^2 f}{\partial (x^1)^2} + \frac{1}{2} \frac{\partial^2 f}{\partial (x^2)^2} = -\frac{2ac^2}{k}. \quad (56)$$

4.3.3 Comparison of monochromatic metric–affine and Einstein–Weyl solutions

The main difference between the two models is that in the metric–affine model our generalised pp-wave solutions have parallel Ricci curvature, whereas in the Einstein–Weyl model the pp-wave type solutions do not necessarily have parallel Ricci curvature. However, when we look at monochromatic pp-wave type solutions in the Einstein–Weyl model their Ricci curvature also becomes parallel. The only remaining difference is in the right-hand sides of equations (54) and (56): in (54) the constant is arbitrary whereas in (56) the constant is expressed via the characteristics of the spinor wave and the gravitational constant.

In other words, comparing equations (54) and (56) we see that while in the metric–affine case the Laplacian of f can be *any* constant, in the Einstein–Weyl case it is required to be a *particular* constant. This should not be surprising as our metric–affine model is conformally invariant, while the Einstein–Weyl model is not.

We also want to clarify that f and the quantities a, b, c appearing in this section 4.3 are generally arbitrary functions of the null coordinate x^3 . As such, if these quantities are non-zero only for a short finite interval of x^3 , the solutions represent spinors, curvature and torsion components which propagate at the speed of light.

Hence we can conclude that generalised pp-waves of parallel Ricci curvature are very similar to pp-type solutions of the Einstein–Weyl model, which is a classical model describing the interaction of massless neutrino and gravitational fields. Therefore we suggest that

*Generalised pp-waves of parallel Ricci curvature represent a metric–affine model
for the massless neutrino.*

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A Spinor Formalism

This appendix provides the spinor formalism used throughout our work. Unless otherwise stated, we work in a general metric compatible spacetime with torsion. When introducing our spinor formalism, we were faced with the problem that there doesn't seem to exist a uniform convention in the existing literature on how to treat spinors. Optimally, we would have wanted to achieve the following:

- (i) consecutive raising and lowering of a spinor index does not change the sign of a rank 1 spinor;
- (ii) the metric spinor ε^{ab} is the raised version of ε_{ab} and vice versa;
- (iii) the spinor inner product is invariant under raising and lowering of indices, i.e. $\xi_a \eta^a = \xi^a \eta_a$.

Unfortunately, it becomes clear that it is not possible to satisfy all three desired properties, as shown in [59]. This inconsistency is related to the well known fact (see for example Section 19 in [6] or Section 3–5 in [65]), that a spinor does not have a particular sign – for example, a spatial rotation of the coordinate system by 2π leads to a change of sign. Also see [56] for more helpful insight about the problem of choice of the spinor formalism, as well as e.g. [6, 7, 12, 29, 63, 59] for insight to various approaches to spinor formalism.

We decided to define our spinor formalism in the following way. We define the ‘metric spinor’ as

$$\varepsilon_{ab} = \varepsilon_{\dot{a}\dot{b}} = \varepsilon^{ab} = \varepsilon^{\dot{a}\dot{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (57)$$

with the first index enumerating rows and the second enumerating columns. We raise and lower spinor indices according to the formulae

$$\xi^a = \varepsilon^{ab} \xi_b, \quad \xi_a = \varepsilon_{ab} \xi^b, \quad \eta^{\dot{a}} = \varepsilon^{\dot{a}\dot{b}} \eta_{\dot{b}}, \quad \eta_{\dot{a}} = \varepsilon_{\dot{a}\dot{b}} \eta^{\dot{b}}. \quad (58)$$

Our definition (57), (58) has the following advantages:

- The spinor inner product is invariant under the operation of raising and lowering of indices, i.e. $(\varepsilon_{ac} \xi^c)(\varepsilon^{ad} \eta_d) = \xi^a \eta_a$.
- The ‘contravariant’ and ‘covariant’ metric spinors are ‘raised’ and ‘lowered’ versions of each other, i.e. $\varepsilon^{ab} = \varepsilon^{ac} \varepsilon_{cd} \varepsilon^{bd}$ and $\varepsilon_{ab} = \varepsilon_{ac} \varepsilon^{cd} \varepsilon_{bd}$.

The disadvantage of our definition (57), (58) is that the consecutive raising and lowering of a single spinor index leads to a change of sign, i.e. $\varepsilon_{ab} \varepsilon^{bc} \xi_c = -\xi_a$. In formulae where the sign is important we will be careful in specifying our choice of sign; see, for example, (59), (63). We in a sense intentionally ‘sacrificed’ this property in order to guarantee that the other two properties, which in our view have greater physical significance, are satisfied.

Let \mathfrak{v} be the real vector space of Hermitian 2×2 matrices σ_{ab} . Pauli matrices σ_{ab}^α , $\alpha = 0, 1, 2, 3$, are a basis in \mathfrak{v} satisfying $\sigma_{ab}^\alpha \sigma^{bc} + \sigma_{ab}^\beta \sigma^{\alpha c} = 2g^{\alpha\beta} \delta_a^c$ where

$$\sigma^{\alpha ab} := \varepsilon^{ac} \sigma_{cd}^\alpha \varepsilon^{bd}. \quad (59)$$

At every point of the manifold M Pauli matrices are defined uniquely up to a Lorentz transformation. Define

$$\sigma_{\alpha\beta ac} := \frac{1}{2} (\sigma_{\alpha ab} \varepsilon^{bd} \sigma_{bcd} - \sigma_{\beta ab} \varepsilon^{bd} \sigma_{acd}). \quad (60)$$

These ‘second order Pauli matrices’ are polarized, i.e.

$$* \sigma = \pm i \sigma \quad (61)$$

depending on the orientation of ‘basic’ Pauli matrices σ_{ab}^α , $\alpha = 0, 1, 2, 3$.

We define the covariant derivatives of spinor fields as

$$\begin{aligned} \nabla_\mu \xi^a &= \partial_\mu \xi^a + \Gamma^a_{\mu b} \xi^b, & \nabla_\mu \xi_a &= \partial_\mu \xi_a - \Gamma^b_{\mu a} \xi_b, \\ \nabla_\mu \eta^{\dot{a}} &= \partial_\mu \eta^{\dot{a}} + \bar{\Gamma}^{\dot{a}}_{\mu b} \eta^{\dot{b}}, & \nabla_\mu \eta_{\dot{a}} &= \partial_\mu \eta_{\dot{a}} - \bar{\Gamma}^b_{\mu \dot{a}} \eta_b, \end{aligned}$$

where $\bar{\Gamma}^{\dot{a}}_{\mu b} = \overline{\Gamma^a_{\mu b}}$. The explicit formula for the spinor connection coefficients $\Gamma^a_{\mu b}$ can be derived from the following two conditions:

$$\nabla_\mu \varepsilon^{ab} = 0, \quad \nabla_\mu j^\alpha = \sigma_{ab}^\alpha \nabla_\mu \zeta^{ab}, \quad (62)$$

where ζ is an arbitrary rank 2 mixed spinor field and $j^\alpha := \sigma^\alpha_{ab} \zeta^{ab}$ is the corresponding vector field (current). Conditions (62) give a system of linear algebraic equations for $\text{Re}\Gamma^a_{\mu b}$, $\text{Im}\Gamma^a_{\mu b}$ the unique solution of which is

$$\Gamma^a_{\mu b} = \frac{1}{4} \sigma^\alpha_{ac} \left(\partial_\mu \sigma^\alpha_{bc} + \Gamma^\alpha_{\mu\beta} \sigma^\beta_{bc} \right). \quad (63)$$

See section 3 of [24] for more background on covariant differentiation of spinors.

B Massless Dirac Equation

The generally accepted point of view [29,30,31,32,35] is that a massless neutrino field is a metric compatible spacetime with or without torsion, described by the action

$$S_{\text{neutrino}} := 2i \int \left(\zeta^a \sigma^\mu_{ab} (\nabla_\mu \bar{\zeta}^b) - (\nabla_\mu \zeta^a) \sigma^\mu_{ab} \bar{\zeta}^b \right), \quad (64)$$

see formula (11) of [29]. We first vary the action (64) with respect to the spinor ζ , while keeping torsion and the metric *fixed*. A straightforward calculation produces the *massless Dirac (or Weyl's)* equation

$$\sigma^\mu_{ab} \nabla_\mu \zeta^a - \frac{1}{2} T^\eta_{\eta\mu} \sigma^\mu_{ab} \zeta^a = 0, \quad (65)$$

which can be equivalently rewritten as

$$\sigma^\mu_{ab} \{\nabla\}_\mu \zeta^a \pm \frac{i}{4} \varepsilon_{\alpha\beta\gamma\delta} T^{\alpha\beta\gamma} \sigma^\delta_{ab} \zeta^a = 0. \quad (66)$$

B.1 Energy momentum tensor

In this subsection we give the derivation of the energy momentum tensor of the action S_{neutrino} , where we vary the metric keeping the spinor fixed. The covariant and contravariant metric change in the following way

$$g_{\alpha\beta} \mapsto g_{\alpha\beta} + \delta g_{\alpha\beta}, \quad g^{\alpha\beta} \mapsto g^{\alpha\beta} - g^{\alpha\alpha'} (\delta g_{\alpha'\beta'}) g^{\beta\beta'}, \quad (67)$$

while the Pauli matrices transform in the following way

$$\sigma_\alpha \mapsto \sigma_\alpha + \frac{1}{2} \delta g_{\alpha\beta} g^{\beta\gamma} \sigma_\gamma, \quad \sigma^\alpha \mapsto \sigma^\alpha - \frac{1}{2} g^{\alpha\beta} (\delta g_{\beta\gamma}) \sigma^\gamma. \quad (68)$$

Formulae describe a 'symmetric' variation of the Pauli matrices caused by the (symmetric) variation of the (symmetric) metric.

Remark 10 We do most of the following calculations under the assumption that the metric is the Minkowski metric $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and that the connection is Levi-Civita.

Now we need to look at the $\delta\Gamma^\alpha_{\beta\gamma}$. Using the definition of the Levi-Civita connection, equation (67) and metric compatibility ($\nabla g \equiv 0$), we get that the connection transforms as

$$\delta\Gamma^\kappa_{\mu\nu} = \frac{1}{2} g^{\kappa\lambda} (\nabla_\mu \delta g_{\lambda\nu} + \nabla_\nu \delta g_{\lambda\mu} - \nabla_\lambda \delta g_{\mu\nu}). \quad (69)$$

Lemma 3 *The variation of the covariant derivative of ξ with respect to the metric is*

$$\delta\nabla_\mu \xi^a = \frac{1}{8} (\sigma^\alpha_{ad} \sigma^\beta_{cd} - \sigma^{\beta ad} \sigma_{\alpha cd}) \xi^c \delta\Gamma^\alpha_{\mu\beta}. \quad (70)$$

Proof Using equation (63), the fact that ξ does not contribute to the variation and the assumptions in Remark 10, we obtain

$$4\delta\nabla_\mu\xi^a = \sigma_\alpha^{ad} \left(\partial_\mu(\delta\sigma_{cd}^\alpha) + (\delta\Gamma^\alpha_{\mu\beta})\sigma_{cd}^\beta \right) \xi^c.$$

Using equation (67) and metric compatibility we get that

$$\partial_\mu(\delta\sigma_{cd}^\alpha) = -\frac{1}{2}g^{\alpha\eta}\sigma_{c\eta d}^\alpha \delta\Gamma^\zeta_{\mu\eta} - \frac{1}{2}\delta^\alpha_\zeta \sigma_{cd}^\zeta \delta\Gamma^\zeta_{\mu\xi}.$$

Combining this with the formula for the variation of $\nabla\xi$, we get the equivalent to equation (70)

$$4\delta\nabla_\mu\xi^a = -\frac{1}{2}\sigma^{\beta ad}\sigma_{\alpha cd}^\alpha \xi^c \delta\Gamma^\alpha_{\mu\beta} - \frac{1}{2}\sigma_\alpha^{adi}\sigma_{cd}^\beta \xi^c \delta\Gamma^\alpha_{\mu\beta} + \sigma_{cd}^\beta \sigma_\alpha^{adi}\xi^c \delta\Gamma^\alpha_{\mu\beta}.$$

□

We now combine equations (69) and (70) to get

$$\delta\nabla_\mu\xi^a = \frac{1}{16}(\sigma^{\lambda ad}\sigma_{cd}^\beta - \sigma^{\beta ad}\sigma_{cd}^\lambda)\xi^c (\partial_\mu\delta g_{\lambda\beta} + \partial_\beta\delta g_{\lambda\mu} - \partial_\lambda\delta g_{\mu\beta}).$$

As the first derivative is symmetric over λ, β and the Pauli matrices are antisymmetric over these indices, we get

$$(\sigma^{\lambda ad}\sigma_{cd}^\beta - \sigma^{\beta ad}\sigma_{cd}^\lambda)\partial_\mu\delta g_{\lambda\beta} = -(\sigma^{\lambda ad}\sigma_{cd}^\beta - \sigma^{\beta ad}\sigma_{cd}^\lambda)\partial_\mu\delta g_{\beta\lambda} = 0.$$

Hence,

$$\delta\nabla_\mu\xi^a = \frac{1}{16}(\sigma^{\lambda ad}\sigma_{cd}^\beta - \sigma^{\beta ad}\sigma_{cd}^\lambda)\xi^c \partial_\beta\delta g_{\lambda\mu} - \frac{1}{16}(\sigma^{\beta ad}\sigma_{cd}^\lambda - \sigma^{\lambda ad}\sigma_{cd}^\beta)\xi^c \partial_\beta\delta g_{\mu\lambda}.$$

So finally, we get the formula for the variation of the covariant derivative of ξ :

$$\delta\{\nabla\}_\mu\xi^a = \frac{1}{8}\xi^c (\sigma^{\alpha ad}\sigma_{cd}^\beta - \sigma^{\beta ad}\sigma_{cd}^\alpha)\partial_\beta\delta g_{\mu\alpha}. \quad (71)$$

Lemma 4 *The energy momentum tensor of the action (64) is equation (48).*

Proof Varying the action (64) with respect to the metric, we get

$$\begin{aligned} \delta S &= 2i\delta \int \left(\xi^a \sigma_{ab}^\eta (\{\nabla\}_\eta \bar{\xi}^b) - (\{\nabla\}_\eta \xi^a) \sigma_{ab}^\eta \bar{\xi}^b \right) \sqrt{|\det g|} \\ &= 2i \int \xi^a (\delta\sigma_{ab}^\eta) (\{\nabla\}_\eta \bar{\xi}^b) + \xi^a \sigma_{ab}^\eta (\delta\{\nabla\}_\eta \bar{\xi}^b) - (\delta\{\nabla\}_\eta \xi^a) \sigma_{ab}^\eta \bar{\xi}^b - (\{\nabla\}_\eta \xi^a) (\delta\sigma_{ab}^\eta) \bar{\xi}^b \\ &\quad + \frac{1}{2} \left(\xi^a \sigma_{ab}^\eta (\{\nabla\}_\eta \bar{\xi}^b) - (\{\nabla\}_\eta \xi^a) \sigma_{ab}^\eta \bar{\xi}^b \right) g^{\mu\nu} \delta g_{\mu\nu} \end{aligned}$$

and using equation (68) we get

$$\begin{aligned} \delta S &= 2i \int \left(-\frac{1}{4}\xi^a g^{\eta\mu} \sigma_{ab}^\nu (\{\nabla\}_\eta \bar{\xi}^b) - \frac{1}{4}\xi^a g^{\eta\nu} \sigma_{ab}^\mu (\{\nabla\}_\eta \bar{\xi}^b) + \frac{1}{4}(\{\nabla\}_\eta \xi^a) g^{\eta\mu} \sigma_{ab}^\nu \bar{\xi}^b \right. \\ &\quad \left. + \frac{1}{4}(\{\nabla\}_\eta \xi^a) g^{\eta\nu} \sigma_{ab}^\mu \bar{\xi}^b + \frac{1}{2}\xi^a \sigma_{ab}^\eta (\{\nabla\}_\eta \bar{\xi}^b) g^{\mu\nu} - \frac{1}{2}(\{\nabla\}_\eta \xi^a) \sigma_{ab}^\eta \bar{\xi}^b g^{\mu\nu} \right) \delta g_{\mu\nu} \\ &\quad + \xi^a \sigma_{ab}^\eta (\delta\{\nabla\}_\eta \bar{\xi}^b) - (\delta\{\nabla\}_\eta \xi^a) \sigma_{ab}^\eta \bar{\xi}^b. \end{aligned}$$

Now we look at the terms involving the variation of $\{\nabla\}_\eta \xi$ on their own. Using equation (71) we get

$$I_1 = \frac{i}{4} \int \xi^a \sigma_{ab}^\mu \bar{\xi}^j (\sigma^{\nu cd}\sigma_{cd}^\eta - \sigma^{\eta cd}\sigma_{cd}^\nu) \partial_\eta \delta g_{\mu\nu} - \xi^c (\sigma^{\nu ad}\sigma_{cd}^\eta - \sigma^{\eta ad}\sigma_{cd}^\nu) \partial_\eta \delta g_{\mu\nu} \sigma_{ab}^\mu \bar{\xi}^b.$$

Integrating by parts and using the simplifications from Remark 10, we get

$$I_1 = \frac{i}{4} \int \bar{\xi}^b \{ \nabla \} \eta \xi^a \left(-\sigma^\mu{}_{ad} \sigma^{bcd} \sigma^\eta{}_{cb} + \sigma^\mu{}_{ad} \sigma^{\eta cd} \sigma^v{}_{cb} + \sigma^\eta{}_{ad} \sigma^{vcd} \sigma^\mu{}_{cb} - \sigma^v{}_{ad} \sigma^{\eta cd} \sigma^\mu{}_{cb} \right. \\ \left. - \sigma^\mu{}_{ad} \sigma^{vcd} \sigma^\eta{}_{cb} + \sigma^\mu{}_{ad} \sigma^{\eta cd} \sigma^v{}_{cb} + \sigma^\eta{}_{ad} \sigma^{vcd} \sigma^\mu{}_{cb} - \sigma^v{}_{ad} \sigma^{\eta cd} \sigma^\mu{}_{cb} \right) \delta g_{\mu\nu}.$$

Since we have $(\sigma^\mu{}_{ad} \sigma^{\eta cd} \sigma^v{}_{cb} - \sigma^v{}_{ad} \sigma^{\eta cd} \sigma^\mu{}_{cb}) \delta g_{\mu\nu} = 0$, as it is a product of symmetric and antisymmetric tensors, as well as (after a lengthy but straightforward calculation)

$$\sigma^\eta{}_{ad} \sigma^{vcd} \sigma^\mu{}_{cb} + \sigma^\eta{}_{ad} \sigma^{\mu cd} \sigma^v{}_{cb} - \sigma^\mu{}_{ad} \sigma^{vcd} \sigma^\eta{}_{cb} - \sigma^v{}_{ad} \sigma^{\mu cd} \sigma^\eta{}_{cb} = 0,$$

we have shown that the terms involving $\delta \{ \nabla \} \xi$ do not contribute to the variation, i.e. $I_1 = 0$. We now return to the variation of the whole action, which after some simplification becomes

$$\frac{\delta S}{\delta g} = \frac{i}{2} \int \left(\sigma^v{}_{ab} (\{ \nabla \} \eta \xi^a \bar{\xi}^b - \xi^a \{ \nabla \} \eta \bar{\xi}^b) + \sigma^\mu{}_{ab} (\{ \nabla \}^v \xi^a \bar{\xi}^b - \xi^a \{ \nabla \}^v \bar{\xi}^b) \right) \delta g_{\mu\nu} \\ + i \int \left(\xi^a \sigma^\eta{}_{ab} (\{ \nabla \} \eta \bar{\xi}^b) g^{\mu\nu} - (\{ \nabla \} \eta \xi^a) \sigma^\eta{}_{ab} \bar{\xi}^b g^{\mu\nu} \right) \delta g_{\mu\nu}.$$

Finally we can conclude that the energy momentum tensor of the action (64) is exactly equation (48). \square

C Correction of explicit form of the second field equation and future work

In calculating the Bianchi identity for curvature in Appendix B from [53], there was an arithmetic error, hence equation (B.11) from that work should read

$$\nabla_\eta Ric^\eta{}_\lambda = -\frac{1}{2} Ric^\eta{}_\xi T_\eta{}^\xi{}_\lambda - \frac{1}{2} \mathcal{W}^{\eta\zeta}{}_{\lambda\xi} (T_\eta{}^\xi{}_\zeta - T_\zeta{}^\xi{}_\eta),$$

and from here equation (B.12) should read

$$\nabla_\eta \mathcal{W}^{\eta}{}_{\mu\lambda\kappa} = \mathcal{W}^{\eta}{}_{\mu\kappa\xi} (T_\eta{}^\xi{}_\lambda - T_\lambda{}^\xi{}_\eta) + \mathcal{W}^{\eta}{}_{\mu\lambda\xi} (T_\kappa{}^\xi{}_\eta - T_\eta{}^\xi{}_\kappa) \\ + \frac{1}{4} (T_\zeta{}^\xi{}_\eta - T_\eta{}^\xi{}_\zeta) (g_{\mu\lambda} \mathcal{W}^{\eta\zeta}{}_{\kappa\xi} - g_{\mu\kappa} \mathcal{W}^{\eta\zeta}{}_{\lambda\xi}) + \frac{1}{4} Ric^\eta{}_\xi (g_{\mu\lambda} T_\eta{}^\xi{}_\kappa - g_{\mu\kappa} T_\eta{}^\xi{}_\lambda) \\ + \frac{1}{2} [\nabla_\lambda Ric_{\mu\kappa} - \nabla_\kappa Ric_{\mu\lambda} + Ric^\eta{}_\kappa (T_{\lambda\eta\mu} - T_{\eta\lambda\mu}) + Ric^\eta{}_\lambda (T_{\eta\kappa\mu} - T_{\kappa\eta\mu})]$$

Consequently, when these two are used in calculating the explicit form of the field equation (3) this produces a different result to the one presented in [53]. Namely, equation (3) in its explicit form (i.e. equation (27) from [53]) should read

$$d_6 \nabla_\lambda Ric_{\kappa\mu} - d_7 \nabla_\kappa Ric_{\lambda\mu} \\ + d_6 \left(Ric^\eta{}_\kappa (T_{\eta\mu\lambda} - T_{\lambda\mu\eta}) + \frac{1}{2} g_{\mu\lambda} \mathcal{W}^{\eta\zeta}{}_{\kappa\xi} (T_\eta{}^\xi{}_\zeta - T_\zeta{}^\xi{}_\eta) + \frac{1}{2} g_{\mu\lambda} Ric^\eta{}_\xi T_\eta{}^\xi{}_\kappa \right) \\ + d_7 \left(Ric^\eta{}_\lambda (T_{\eta\mu\kappa} - T_{\kappa\mu\eta}) + \frac{1}{2} g_{\kappa\mu} \mathcal{W}^{\eta\zeta}{}_{\lambda\xi} (T_\eta{}^\xi{}_\zeta - T_\zeta{}^\xi{}_\eta) + \frac{1}{2} g_{\kappa\mu} Ric^\eta{}_\xi T_\eta{}^\xi{}_\lambda \right) \\ + b_{10} (T_\eta{}^\xi{}_\zeta - T_\zeta{}^\xi{}_\eta) (g_{\mu\kappa} \mathcal{W}^{\eta\zeta}{}_{\lambda\xi} - g_{\mu\lambda} \mathcal{W}^{\eta\zeta}{}_{\kappa\xi}) \\ + 2b_{10} \left(\mathcal{W}^{\eta}{}_{\mu\kappa\xi} (T_\eta{}^\xi{}_\lambda - T_\lambda{}^\xi{}_\eta) + \mathcal{W}^{\eta}{}_{\mu\lambda\xi} (T_\kappa{}^\xi{}_\eta - T_\eta{}^\xi{}_\kappa) - \mathcal{W}^{\xi\eta}{}_{\kappa\lambda} T_{\eta\mu\xi} \right) = 0.$$

This change in no way affects the proof of the main theorem, as this form of the field equation is even simpler and does not contain any additional terms, but we believed it was important to point out for purposes of future work. This mistake was noticed in the process of generalising the explicit form of the field equations (2), (3), i.e. the equations equation (26) and (27) from [53], see [52].

The two papers of Singh [60, 61] are of particular interest to us, where the author constructs solutions for the Yang–Mills case (4) with purely axial and purely trace torsion respectively, see (9), and unlike the solution of [63], $\{Ric\}$ is not assumed to be zero. It is obvious that these solutions differ from the ones presented in our work, as the torsion of generalised pp-waves is assumed to be purely tensor. It would however be of interest to us to see whether this construction of Singh's can be expanded to our most general $O(1,3)$ -invariant quadratic form q with 16 coupling constants.

We also hope to see whether it is possible to produce torsion waves which are purely axial or trace and combine them with the pp-wave metric, in a similar fashion as was done with purely tensor torsion waves, in order to produce new solutions of quadratic metric–affine gravity and also give *their* physical interpretation in the near future.

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